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THE STABILITY OF HAMILTONIAN SYSTEMS IN THE CASE OF A MULTIPLE FOURTH-ORDER RESONANCE†

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The problem of the stability of the position of equilibrium of a multidimensional autonomous Hamiltonian system is studied for the critical case of purely imaginary simple roots when the quadratic part of the Hamiltonian is not sign-definite and the roots satisfy simultaneously two (or more) resonant fourth-order relations [1]. Cases of independent as well as mutually interacting resonances are discussed. The conditions of stability and instability of the corresponding normalized system containing terms of up to the fourth order inclusive, are formulated.

CONSIDER the problem of the stability of the stagnation point of an autonomous system of Hamiltonian equations defined by the Hamiltonian function $H(x, y) = H_2 + H_3 + \ldots$, where H_l are the lth order forms of the variables $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, Y_N)$, under the assumption that the form is not sign-definite and all eigenvalues of its matrix are purely imaginary and differ from each other. As we know [1, 2] the most interesting cases are the resonant ones (when there are no resonances we have complete Birkhof stability [2]) in which instability may occur, caused by the non-linear terms of the corresponding differential equations.

We shall consider the case, which has not so far been studied, of a double, fourth-order resonance governed by the presence of two integer relations between the eigenvalues $\pm\lambda_s$ $(s = 1, \ldots, N)$ of the form

$$\Sigma p_\alpha\lambda_\alpha = 0, \quad \Sigma p_\beta\lambda_\beta = 0$$
$$\Sigma|p_\alpha| = \Sigma|p_\beta| = 4 \quad n < N \tag{1}$$

(p_α, p_β are mutually prime numbers), or of the form

$$p_{11}\lambda_1 + p_2\lambda_2 + \ldots + p_m\lambda_m = 0, \qquad p_{21}\lambda_1 + \Sigma p_\beta\lambda_\beta = 0$$
$$|p_{11}| + |p_2| + \ldots + |p_m| = |p_{21}| + \Sigma|p_\beta| = 4 \tag{2}$$

Here and henceforth the summation over α will be carried out from $\alpha = 1$ to $\alpha = m$, over β from $\beta = m+1$ to $\beta = n$, and over γ from $\gamma = n+1$ to $\gamma = N$.

We shall also assume that relations (1) and (2) will not produce other resonances of the same order.

It is said that in case (1) the resonances are independent, and in case (2) we have interaction between the resonances. Moreover, we shall call a resonance weak if it preserves, when there are no other resonances, the stability of the model system (i.e. of the system obtained by discarding, from H, terms of order higher than the fourth). Otherwise, we shall call the resonance strong [4, 5].

Let us consider the case of independent resonances (1). Using the well-known normalization procedure [1–3] we shall obtain the following form of a Hamiltonian normalized (to terms of up to the fourth order) in polar coordinates r_j, θ_j:

$$H = \Sigma\lambda_j r_j + H_4, \quad H_4 = 2\sum_{\nu=1}^{2} A_\nu\sqrt{R_\nu}\cos\Psi_\nu + \Sigma A^{lj} r_l r_j$$
$$R_1 = \Pi r_\alpha^{|p_\alpha|}, \quad R_2 = \Pi r_\beta^{|p_\beta|}, \quad \Psi_1 = \Sigma p_\alpha\theta_\alpha, \quad \Psi_2 = \Sigma p_\beta\theta_\beta \tag{3}$$

Here and henceforth the subscripts l and j take all values from 1 to N, and the indices α, β, γ have the same values as in (1) and (2).

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The model system corresponding to the Hamiltonian (3) will have the form

$$r_\alpha^\cdot = -2p_\alpha A_1\sqrt{R_1}\sin\Psi_1,\quad r_\beta^\cdot = -2p_\beta A_2\sqrt{R_2}\sin\Psi_2,\quad r_\gamma^\cdot = 0 \tag{4}$$

$$\psi_1^\cdot = -A_1\sqrt{R_1}\,\Sigma p_\alpha^2/r_\alpha\cos\Psi_1 - 2\Sigma A^{\alpha j}p_\alpha r_j$$

$$\psi_2^\cdot = -A_2\sqrt{R_2}\,\Sigma p_\beta^2/r_\beta\cos\Psi_2 - 2\Sigma A^{\beta j}p_\beta r_j$$

We shall show that the following theorem holds for system (4) in case (1).

Theorem 1. If amongst the resonances of (1) there exists at least one strong resonance, then the trivial solution of system (4) will be unstable.

Let the variables r_α in (4) correspond to a strong resonance. Then assuming all $r_\beta = 0$ we arrive at a system of equations describing a situation with a single resonance which, according to the assumption made above, leads to instability.

We note that the condition of Theorem 1 remains true for any number of resonances.

Let us now consider the case in which both resonances (1) are weak. Here we must distinguish between two types of weak resonance, namely: (a) the weakness of the resonance depends on the sign change amongst the components of the resonant vector $P = (p_1, \ldots, p_m)$; (b) all p_α and p_β are of the same sign and the weakness of each resonance is governed by the inequalities [3, 6]

$$|A_\nu| < |S_\nu| \quad (\nu = 1, 2) \tag{5}$$

$$S_1 = \frac{1}{2P_1}\sum_{\alpha,\delta=1}^{m} A^{\alpha\delta}p_\alpha p_\delta,\quad S_2 = \frac{1}{2P_2}\sum_{\beta,\sigma=m+1}^{n} A^{\beta\sigma}p_\beta p_\sigma \tag{6}$$

$$P_1 = \Pi p_\alpha^{p_\alpha/2},\quad P_2 = \Pi p_\beta^{p_\beta/2}$$

In this case the following theorem holds.

Theorem 2. If both independent resonances of (1) are weak and at least one of them is weak in the sense of (a), then the trivial solution of the model system is stable.

We shall first consider the case when both resonances of (1) are weak in the sense (a), i.e. the sign change occurs amongst the components $p_1, \ldots, p_m$, as well as amongst $p_{m+1}, \ldots, p_n$. Then the system will have the integral

$$\Phi = \Sigma\gamma_\alpha r_\alpha + \Sigma\gamma_\beta r_\beta + \Sigma r_\gamma$$

(γ_α, γ_β are certain constants), which is sign definite under the conditions of the theorem. Indeed, the requirement that the derivative $\Phi^\bullet$ vanishes identically leads, for system (4), to the equations

$$\Sigma\gamma_\alpha p_\alpha = 0,\quad \Sigma\gamma_\beta p_\beta = 0$$

which always have a strictly positive solution in γ_α, γ_β, provided that there is a change of sign amongst the numbers $p_1, \ldots, p_m$ and $p_{m+1}, \ldots, p_n$.

Suppose now that only one of the resonances of (1) is weak in the sense (a) we can assume without loss of generality that the first resonance is the weak one, while for the second resonance we shall have $|A_2| < |S_2|$. In this case system (4) will have the following integrals:

$$\Phi = \Sigma\gamma_\alpha r_\alpha + \Sigma r_\gamma,\quad I_\beta = r_\beta - (p_\beta/p_{m+1})r_{m+1},\quad H_* = H_4$$

from which we shall construct the integral

$$G = \Phi^4 + I_2^4 + \ldots + I_n^4 + H_*^2$$

which is sign-definite. Indeed, since we have $\Phi = I_\beta = 0$ when $r_\alpha = r_\gamma = 0$ and $r_\beta = (p_\beta/p_{m+1})r_{m+1}$, it follows that

$$H_4 = 2(A_2\cos\Psi_2 + S_2)(r_{m+1}^2/p_{m+1}^2)P_2$$

whence, taking into account the inequality $|A_2| < |S_2|$, we find that G is a positive-definite function.

The case when both resonances of (1) are weak in the sense (b), i.e. inequality (5) holds for each one of them, is more complicated. Let us introduce the notation

$$S' = \Sigma A^{\alpha\beta}p_\alpha p_\beta,\quad S_1' = \frac{S}{2P_1},\quad S_2' = \frac{S'}{2P_2} \tag{7}$$

and consider the following versions of the signs of the quantities (6) and (7):

1. S_1, S_2, S' are of the same sign.
2. S_1, S_2 are of the same sign and S' is of the opposite sign.
3. S_1, S_2 are of different sign.

The following assertion holds in the first case.

Theorem 3. If the model system (4) has two weak resonances, in the sense (b), and there is no change of sign amongst the quantities S_1, S_2, S', then the trivial solution of system (4) will be stable.

It can be confirmed that in this case system (4) will have the follwing integrals:

$$I_\delta = r_\delta - \frac{p_\delta}{p_1} r_1, \quad I_\sigma = r_\sigma - \frac{p_\sigma r_{m+1}}{p_{m+1}}, \quad \Phi = \Sigma r_\gamma, \quad H_* = H_4$$
$$(\delta = 2, \ldots, m; \quad \sigma = m+2, \ldots, n)$$

from which we can construct a sign-definite integral in the form

$$G = \Sigma I_\delta^2 + \Sigma I_\sigma^2 + H_*^2 \tag{8}$$

Indeed, let

$$r_\alpha = (p_\alpha/p_1) r_1, \quad r_\beta = (p_\beta/p_{m+1}) r_{m+1}, \quad r_\gamma = 0$$

then we have $I_\delta = I_\sigma = \Phi = 0$ and

$$H_* = 2(A_1 \cos\Psi_1 + S_1)(r_1^2/p_1^2) P_1 + 2S'(r_1 r_{m+1}/p_1 p_{m+1}) + \\ + 2(A_2 \cos\Psi_2 + S_2)(r_{m+1}^2/p_{m+1}^2) P_2 \tag{9}$$

Thus under the condition of Theorem 3 the form does not vanish, except at the origin of coordinates.

Let us now consider the second case when S_1 and S_2 are of the same sign, and S' is of the opposite sign. Two subcases are possible:

$$1)\ S_1 S_2 > S_1' S_2'; \quad 2)\ S_1 S_2 < S_1' S_2' \tag{10}$$

Theorem 4. Let two resonances exist in system (4), weak in the sense (b) $S_1S_2>0$, $S_1S'<0$. Then the necessary and sufficient condition of stability of the trivial solution of system (4) under the first condition of (10) will be that the following inequalities hold:

$$|S_1 - A_1|/|S_1'| > |S_2'|/|S_2 - A_2| \tag{11}$$

We shall prove the sufficiency with help of the integrals of Theorem 3. Let us rewrite expression (9) in the form

$$H_* = [2(A_2 \cos\Psi_2 + S_2)\frac{p_1^2 r_{m+1}^2}{p_{m+1}^2 r_1^2} P_2 + 2S' \frac{p_1 r_{m+1}}{p_{m+1} r_1} + \\ + 2(A_1 \cos\Psi_1 + S_1) P_1]\frac{r_1^2}{p_1^2}$$

When condition (11) holds, the discriminant of the equation $H_* = 0$ in $p_1 r_{m+1}/(p_{m+1} r_1)$ is negative, and this leads to sign definiteness of the integral (8).

To prove the necessity we shall consider the opposite inequality to (11) (we note that this can always happen if the first condition of (10) holds with the equality sign). Then it can be shown that system (4) has a particular solution of the form

$$r_\alpha = p_\alpha b(t), \quad r_\beta = p_\beta A_1 P_1 \sin\Psi_1^*/(P_2 A_2 \sin\Psi_2^*)$$
$$b(t) > 0, \quad \dot b(t) > 0, \quad \Psi_1^* = \text{const}, \quad \Psi_2^* = \text{const}$$

which proves the instability.

We prove, in exactly the same manner, the instability in system (4) in the case of the second condition of (10), when the following inequality holds:

$$|S_1 + A_1|/|S_1'| > |S_2'|/|S_2 + A_2|$$

A complete discussion of this subcase, as well as of the case when S_1 and S_2 have different signs, requires methods different from those given here.

We will now consider the case of interacting resonances locked into a single frequency so that relation (2) holds. The model system here takes the form (here and henceforth the index δ takes all natural values from 2 to m)

$$
\begin{aligned}
&\dot r_1 = -2p_{11}A_1\sqrt{R_1}\sin\Psi_1 - 2p_{21}A_2\sqrt{R_2}\sin\Psi_2\\
&\dot r_\delta = -2p_\delta A_1\sqrt{R_1}\sin\Psi_1, \quad \dot r_\beta = -2p_\beta A_2\sqrt{R_2}\sin\Psi_2\\
&\dot r_\gamma = 0\\
&\dot\Psi_1 = -A_1\sqrt{R_1}(p_{11}^2/r_1 + \Sigma p_\delta^2/r_\delta)\cos\Psi_1 - A_2\sqrt{R_2}(p_{11}p_{21}/r_1)\cos\Psi_2 -\\
&- 2p_{11}\Sigma A^{1j}r_j - 2\Sigma A^{\delta j}p_\delta r_j\\
&\dot\psi_2 = -A_2\sqrt{R_2}(p_{21}^2/r_1 + \Sigma p_\beta^2/r_\beta)\cos\Psi_2 -\\
&- A_1\sqrt{R_1}(p_{11}p_{22}/r_1)\cos\Psi_1 - 2p_{21}\Sigma A^{1j}r_j - 2\Sigma A^{\beta j}p_\beta r_j\\
&R_1 = r_1^{|p_{11}|}\Pi r_\delta^{|p_\delta|}, \quad R_2 = r_1^{|p_{21}|}\Pi r_\beta^{|p_\beta|}\\
&\Psi_1 = p_{11}\theta_1 + \Sigma p_\delta\theta_\delta, \quad \Psi_2 = p_{21}\theta_1 + \Sigma p_\beta\theta_\beta
\end{aligned}
\tag{12}
$$

Theorem 5. If at least one strong resonance exists amongst the resonances of (2), the trivial solution of system (12) will be unstable.

The proof is exactly the same as in the case of Theorem 1, and covers any number of resonances with a single common frequency.

In the case of interacting weak resonance with a single common frequency, the situation becomes more complicated than in the case of independent resonances. In this case the sufficient conditions of stability are given by the following theorem.

Theorem 6. If both resonances of (2) are weak in the sense (a), then the trivial solution of system (12) will be stable; if, on the other hand, one of the resonances, e.g. the first resonance, is weak in the sense (a) and the second is weak in the sense (b), then the trivial solution of system (12) will be stable when there is a sign change amongst the numbers $p_2, \ldots, p_m$ in the first resonance.

The first part of the theorem is proved in exactly the same manner as in the case of independent resonances (Theorem 2).

To prove the stability in the case when the second resonance is weak in the sense (b), i.e. when the inequality

$$|S_2| \equiv |(A^{11}p_{21}^2 + 2p_{21}\Sigma A^{1\beta}p_\beta + \Sigma A^{\beta\sigma}p_\beta p_\sigma)/P_2| > |A_2| \tag{13}$$

holds, we shall use the integrals of system (12)

$$
\begin{aligned}
&\Phi = \Sigma\gamma_\delta r_\alpha' \quad (\gamma_\delta > 0) \quad I_\beta = r_\beta - (p_\beta/p_{m+1})r_{l\,l+1}\\
&I_{\alpha\beta} = r_1 - (p_{11}/p_\delta)r_\delta - (p_{21}/p_\beta)r_\beta\\
&H_* = 2\Sigma A_\nu\sqrt{R_\nu}\cos\Psi_\nu + \Sigma A^{ij}r_ir_j\\
&\delta = 2, \ldots, m; \quad \beta = m+2, \ldots, n; \quad \nu = 1, 2; \quad i, j = 1, \ldots, N
\end{aligned}
$$

to construct the integral

$$G = \Sigma I_{\delta\beta}^4 + \Sigma I_\beta^4 + \Phi^4 + H_*^2,$$

which, according to the condition of the theorem, is sign-definite. Indeed, if the variables vary according to the law $r_\alpha = 0$, $r_\beta = (p_\beta/p_{m+1})r_{m+1}$, $r_1 = (p_{21}/p_{m+1})r_{m+1}$ so that the integral Φ, I_β, $I_{\delta\beta}$ vanishes identically, then according to (13) we shall have

$$H_* = (A_2\cos\Psi_2 + S_2)P_2r_{m+1}^2/p_{m+1}^2$$

and hence G is a positive-definite function.

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THE SIGN-DEFINITE CRITERION OF A HOMOGENEOUS POLYNOMIAL IN A CONE†

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A sign-definite criterion of a polynomial of degree m in a cone $K\{\alpha_{10},\ldots,\alpha_{n0}\}$ of space R^n is proposed and also a method of investigating these properties based on certain results obtained from Sirazetdinov. This enables the solution of the problem of the stability of systems of differential equations with polynomial right-hand sides to be simplified.

1. FORMULATION OF THE PROBLEM

IN CERTAIN problems of stability (for example, in problems in economics, stability in biological societies, etc.), there is no need to use functions with sign-definite properties over the whole of the space R^n. For the systems of ordinary differential equations which describe these processes, a certain set $K\subset R^n$ is positively invariant. The trajectories of the system with initial data from K do not leave its limits as time passes. This set is called a cone, it is closed and all its elements possess the following properties: (1) for any $x\in K$ it follows that $-x\bar{\in} K$ ($x\neq 0$, 0) is zero, and (2) for any α, $\beta>0$ and arbitrary u, $v\in K$ it follows that $\alpha u+\alpha v\in K$.

Henceforth we will consider the case when the cone coincides with the coordinate angle. We will use the notation [1] $K\{\alpha_{10},\ldots,\alpha_{n0}\}$, $\alpha_{i0}\in N_0=\{-1,1\}$. Here $\{\alpha_{i0}\}$, $(i=1,\ldots,n)$ is the basis of the cone K. In this case

$$\alpha_{i0}=\operatorname{sign} x_i\,(x_i\neq 0),\qquad i=1,\ldots,n;\qquad \alpha_{i0}x_i\geqslant 0$$

If the problem involves considering a system of ordinary differential equations whose trajectories do not leave the limits of the cone as time passes, then when solving the problem of the stability of this system there is no need to use as the function a Lyapunov function that is sign-definite over the whole of space. It is sufficient for it to possess this property solely in the cone K.

Hence, the problem arises of investigating the sign-definite properties of different functions, in particular, homogeneous polynomial-forms in a certain cone K of space R^n.

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